Math 247A Lecture 16 Notes

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1 Boundedness of Calderón-Zygmund Convolution Kernels

1.1 L^2-boundedness of convolution with Calderón-Zygmund kernels

Last time, we were proving the following theorem.

Theorem 1.1. *Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K \mathbb{1}_{\{\varepsilon < |x| \leq 1/\varepsilon\}}$. Then*

$$\|K_\varepsilon * f\|_2 \lesssim \|f\|_2$$

uniformly for $\varepsilon > 0, f \in L^2$. Consequently, $f \mapsto K * f$ (which is the L^2 limit as $\varepsilon \to 0$ of $K_\varepsilon * f$) extends continuously from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on $L^2(\mathbb{R}^d)$.

Proof. By Plancherel,

$$\|K_\varepsilon * f\|_{L^2} \leq \|\widehat{K}_\varepsilon\|_\infty \|f\|_{L^2},$$

so it suffices to show that $\|\widehat{K}_\varepsilon\|_\infty \lesssim 1$ uniformly in $\varepsilon > 0$. Fix $\xi \in \mathbb{R}^d$. Then

$$\widehat{K}_\varepsilon(\xi) = \int e^{-2\pi i x \cdot \xi} K_\varepsilon(x)\, dx$$

$$= \int_{|x| \leq 1/|\xi|} + \int_{|x| > 1/|\xi|}$$

Because of property (b) and (a),

$$\begin{aligned}\int_{\varepsilon \leq |x| \leq 1/|\xi|} e^{-2\pi i x \cdot \xi} K_\varepsilon(x)\, dx &= \int_{|x| \leq 1/|\xi||} \left| [e^{-2\pi i x \cdot \xi} - 1] K_\varepsilon(x) \right| \\ &\lesssim \int_{|x| \leq 1/|\xi|} |x| \cdot |\xi| \cdot \frac{1}{|x|^d}\, dx \lesssim 1.\end{aligned}$$

We also have

$$\int_{|x| > 1/|\xi|} e^{-2\pi i x \cdot \xi} K_\varepsilon(x)\, dx = \int_{|x| > 1/|\xi|} \frac{1}{2} (e^{-2\pi i x \cdot \xi} - e^{-2\pi i \xi \cdot (x - \xi/(2|\xi|^2))}) K_\varepsilon(x)\, dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{|x|>1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx \\
&\quad - \frac{1}{2}\int_{x+\xi/(2|\xi|^2)>1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x+\xi/(2\|\xi|^2))\,dx \\
&= \frac{1}{2}\int_{|x|>1/|\xi|} e^{-2\pi i x\cdot\xi} [K_\varepsilon(x) - K_\varepsilon(x+\xi/(2|\xi|^2))]\,dx \\
&\quad + \frac{1}{2}\int e^{-2\pi i x\cdot\xi} K_\varepsilon(x+\xi/(2|\xi|^2))\,dx
\end{aligned}
$$

We can split $\int_{x+\xi/(2|\xi|^2)>1/|\xi|} = \int_{|x|>1/|\xi|} - \int_A - \int_B$, where A and B are a partition of the symmetric difference (like a Venn diagram). So $A = \{x : |x| \leq 1/|\xi| \leq |x+\xi/(2|\xi|^2)|\}$ and $B = \{x : |x+\xi/(2|\xi|^2)| \leq 1/|\xi| \leq |x|\}$.

$$
\begin{aligned}
&= \underbrace{\int_{|x|>1/|\xi|} e^{-2\pi i x\cdot\xi} [K_\varepsilon(x) - K_\varepsilon(x+\xi/(2|\xi|^2))]\,dx}_{I} \\
&\quad \underbrace{- \frac{1}{2}\int_A e^{-2\pi i x\cdot\xi} K_\varepsilon(x+\xi/(2|\xi|^2))\,dx}_{II} \\
&\quad \underbrace{- \frac{1}{2}\int_B e^{-2\pi i x\cdot\xi} K_\varepsilon(x+\xi/(2|\xi|^2))\,dx\,.}_{III}
\end{aligned}
$$

Looking at these terms individually:

$$
|I| \leq \frac{1}{2}\int_{|x|\geq 1/|\xi|} |K_\varepsilon(x) - K_\varepsilon(x+\xi/(2|\xi|^2))|\,dx \lesssim 1
$$

uniformly in ξ and $\varepsilon > 0$ by condition (c).

$$
|II| \lesssim \int_A |K_\varepsilon(x+\xi/(2|\xi|^2))|\,dx
$$

Note that $A \subseteq x : 1/|\xi| \leq |x+\xi|/(2|\xi|^2)| \leq |x| + 1/(2|xi|) \leq 3/(2|\xi|)$.

$$
\begin{aligned}
&\lesssim \int_{1/|\xi|\leq|y|\leq 3/(2|\xi|)} |K_\varepsilon(y)|\,dy \\
&\lesssim \int_{1/|\xi|\leq|y|\leq 3/(2|\xi|)} .
\end{aligned}
$$

$$
|III| \lesssim \int_B |K_\varepsilon(x+\xi/(2|\xi|^2))|\,dx
$$

Note that $B \subseteq x : 1/(2|\xi|) \leq |x| - 1/(2|xi|) \leq |x+\xi|/(2|\xi|^2)| \leq 1/|\xi|$.

$$\lesssim \int_{1/(2|\xi|)\leq|y|\leq 1/|\xi|} |K_\varepsilon(y)|\,dy \lesssim 1.$$

So $\|\widehat{K}_\varepsilon\|_\infty \lesssim 1$, uniformly in $\varepsilon > 0$.

We claim that for $f \in \mathcal{S}(\mathbb{R})$, $\{K_\varepsilon * f\}_\varepsilon$ is Cauchy ni L^2. Assuming the claim, for $f \in \mathcal{S}(\mathbb{R}^d)$, let $K * f$ be the L^2 limit of $K_\varepsilon * f$. Then

$$\|K * f\|_2 \leq \underbrace{\|K_\varepsilon * f\|_2}_{\lesssim\|f\|_2} + \underbrace{\|K * f - K_\varepsilon * f\|_2}_{\xrightarrow{\varepsilon\to 0} 0}.$$

So we have that

$$\|K * f\|_2 \lesssim \|f\|_2 + o(1)$$

as $\varepsilon \to 0$. Let $\varepsilon \to 0$ to get $\|K * f\|_2 \lesssim \|f\|_2$. For $f \in L^2$, let $f_n \in \mathcal{S}$ be such that $f_n \xrightarrow{L^2} f$. Then $\{f_n\}_n$ is Cauchy in L^2, so $\{K * f_n\}_{n\geq 1}$ is Cauchy in L^2. Let $K * f$ be the L^2-limit of $K * f_n$. Now

$$\|K * f\|_2 = \lim_n \|K * f\|_2 \lesssim \lim_n \|f_n\|_2 = \|f\|_2.$$

Now let's prove the claim: Fix $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 < \varepsilon_1 < \varepsilon_2 < 1$. Then

$$\begin{aligned}(K_{\varepsilon_1} * f - K_{\varepsilon_2} * f)(x) &= \int_{\varepsilon_1\leq|y|\leq 1/\varepsilon_1} K(y)f(x-y)\,dy - \int_{\varepsilon_2\leq|y|\leq 1/\varepsilon_2} K(y)f(x-y)\,dy \\ &= \int_{\varepsilon_1\leq|y|\leq\varepsilon_2} K(y)f(x-y)\,dy + \int_{1/\varepsilon_2\leq|y|\leq 1/\varepsilon_1} K(y)f(x-y)\,dy\end{aligned}$$

Using property (b),

$$\begin{aligned}\left|\int_{\varepsilon_1\leq|y|\leq\varepsilon_2} K(y)f(x-y)\,dy\right| &= \left|\int_{\varepsilon_1\leq|y|\leq\varepsilon_2} K(y)[f(x)-f(y)])\,dy\right| \\ &\leq \int_{\varepsilon_1\leq|y|\leq\varepsilon_2} |K(y)||y| \int_0^1 |\nabla f(x-\theta y)|\,d\theta\,dy\end{aligned}$$

Using property (a),

$$\begin{aligned}&\lesssim \int_{\varepsilon_1\leq|y|\leq\varepsilon_2} |y|^{1-d} \int \underbrace{|\nabla f(x-\theta y)|}_{\lesssim 1/\langle x-\theta y\rangle^d \lesssim 1/\langle x\rangle^d}\,d\theta\,dy \\ &\lesssim (\varepsilon_2 - \varepsilon_1)\frac{1}{\langle x\rangle^d}.\end{aligned}$$

Alternatively, we could say

$$\begin{aligned}\left\| \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |y|^{1-d} \int |\nabla f(x-\theta y)| \, d\theta \, dy \right\|_{L^2_x} &\lesssim \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |y|^{1-d} \int \|\nabla f(x-\theta y)\|_{L^2_x} \, d\theta \, dy \\ &\lesssim \|\nabla f\|_{L^2}(\varepsilon_2 - \varepsilon_1) \\ &\xrightarrow{\varepsilon_2, \varepsilon_1 \to 0} 0.\end{aligned}$$

For the other term, using Young's inequality, we have

$$\begin{aligned}\left\| \int_{1/\varepsilon_2 \le |y| \le 1/\varepsilon_1} \int K(y) f(x-y) \, dy \right\|_{L^2_x} &\lesssim \|K \mathbb{1}_{\{1/\varepsilon_2 \le |y| \le 1/\varepsilon_1\}}\|_2 \cdot \|f\|_1 \\ &\lesssim \|f\|_{L^1} \left(\int_{|y| \ge 1/\varepsilon_2} |y|^{-2d} \, dy \right)^{1/2} \\ &\lesssim \|f\|_1 \varepsilon_2^{d/2} \\ &\xrightarrow{\varepsilon_2 \to 0} 0. \qquad \square\end{aligned}$$

Remark 1.1. The same argument show that for $f \in \mathcal{S}(\mathbb{R}^d)$, $\{K_\varepsilon * f\}_{\varepsilon > 0}$ is Cauchy in L^p for $1 < p < \infty$. It uses conditions (a), (b).

1.2 L^p bounds for Calderón-Zygmund convolution kernels

Theorem 1.2. *Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K \mathbb{1}_{\{\varepsilon \le |x| \le 1/\varepsilon\}}$. Then*

1. *$|\{x : |K_\varepsilon * f|(x) > \lambda\}| \lesssim \frac{1}{\lambda} \|f\|_1$ uniformly in $\lambda > 0, f \in L^1, \varepsilon > 0$.*
2. *For any $1 < p < \infty$, $\|K_\varepsilon * f\|_p \lesssim \|f\|_p$ uniformly for $f \in L^p, \varepsilon > 0$.*

Consequently, $f \mapsto K * f$ (the L^p-limit of $K_\varepsilon * f$) extends continuous ly from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on L^p when $1 < p < \infty$.

Proof. First, assume that we have proven the first claim. By the Marcinkiewicz interpolation theorem, we get the second claim for $1 < p < 2$. Now fix $2 < p < \infty$. By duality,

$$\begin{aligned}\|K_\varepsilon * f\|_p &= \sup_{\|g\|_{p'} = 1} \langle K_\varepsilon * f, g \rangle \\ &= \sup_{\|g\|_{p'} = 1} \langle f, \overline{K_\varepsilon^R} * g \rangle \\ &\lesssim \|f\|_p \sup_{\|g\|_{p'} = 1} \|\overline{K_\varepsilon^R} * g\|_{p'} \\ &\lesssim \|f\|_p. \qquad \square\end{aligned}$$

We will prove the first claim last time.